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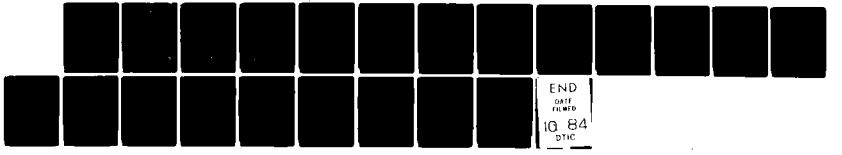

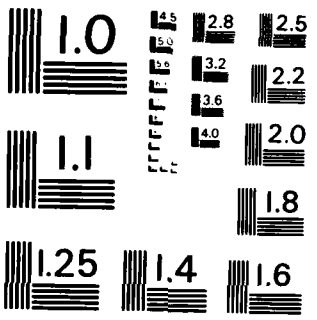

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MELBOURNE, VICTORIA

Aerodynamics Technical Memorandum 363

A COMPARATIVE STUDY OF THE FINITE ELEMENT AND BOUNDARY ELEMENT METHODS AS APPLIED TO A BOUNDARY VALUE PROBLEM OF A HARMONIC FUNCTION

TON TRAN-CONG

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by

TON TRAN-CONG

SUMMARY

The Finite Element and Boundary Element Methods are described with their essential features illustrated using an example of a boundary value problem for a harmonic function. Analysis of the methodical errors is then carried out. This is followed by a consideration of the relative computational advantages of the two methods.

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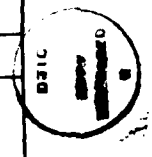

1. INTRODUCTION

In conducting studies of aircraft flight dynamic behaviour it is often necessary to estimate the aerodynamic characteristics. Methods such as those embodied in the USAF Stability and Control DATCOM and in the ESDU data sheets are available, but they are rather empirical with all their inherent limitations. Recent developments in computational fluid dynamics offer the prospect of estimating aerodynamic characteristics directly from physical principles at a cost acceptable for engineering purposes.

As a first step towards adopting such methods, an examination of the relative merits of the Finite Element and the Boundary Element methods is made. The purpose of this Memorandum is to provide a concise and intelligible summary of the two methods, with sufficient emphasis on their essential features. Such a summary is thought to be desirable in the face of a proliferation of bewildering and often fragmented literature on the subject (for example, see the list of references of [1] to [4]). The example chosen to illustrate the methods is the boundary value problem for a harmonic function. This problem is considered typical as such familiar problems as elastostatics and potential flows can all be reduced to the problem of determining harmonic functions.

The analysis of methodical errors is then presented. It is surprising that such essential analysis is not readily accessible in spite of the large amount of literature on the two methods. The claim that the Boundary Element Method is computationally advantageous compared with the Finite Element Method is also examined. This claim is hard to substantiate when the test problem concerns a three dimensional region.

Finally, an appendix proves the convergence of the Gauss-Seidel iteration method for positive definite Hermitian matrices. Such matrices often arise(s) from the use of the Finite Element Method involving complex-valued functions.

2. FORMULATION OF THE FINITE ELEMENT METHOD

This method depends of the existence of a functional I, the extremal value of which gives the required solution to our particular problem [2,3,4]. To fix ideas, we consider the problem of finding a real-valued function $\phi(x)$ which has continuous second order derivatives in a region Ω, $\phi \in C^2(\Omega)$, such that

$$\nabla^2\phi = 0 \quad \text{in the region } \Omega \qquad ..(1)$$

subjected to the boundary conditions

$$\phi = u \quad \text{on } \Gamma_1, \qquad ..(2)$$

$$\frac{\partial \phi}{\partial n} = p_2 \quad \text{on} \quad \Gamma_2 \qquad ..(3)$$

where the two disjoint surfaces Γ_1 and Γ_2 form the boundary Γ of the region Ω. (The region Ω and its boundary Γ are such that all classical theorems of calculus are applicable). It is straightforward to use the divergence theorem to prove that the above problem has only one solution $\phi(\underset{\sim}{x})$. When $\Gamma = \Gamma_1$ the above boundary value problem becomes the Dirichlet problem and when $\Gamma = \Gamma_2$, the Neumann problem (only in this latter case $\phi(\underset{\sim}{x})$ is determined only up to a constant). Here we take for granted that there exists a solution $\phi(x)$ to the above boundary value problem.

Let us form the functional

$$I(\phi) = \int_{\Omega} |\underset{\sim}{\nabla}\phi|^2 - 2 \int_{\Gamma} \phi\, p \qquad ..(4)$$

where $p = p_2$ on Γ_2 and $p = 0$ on Γ_1. Minimization of this functional I with respect to $\phi(\underset{\sim}{x})$ such that $\phi = u$ on Γ_1 gives

$$0 = - \int_{\Omega} \delta\phi\, \nabla^2\phi + \int_{\Gamma} \delta\phi \left(\frac{\partial \phi}{\partial n} - p \right) \qquad ..(5)$$

for any arbitrary incremental function $\delta\phi \in C^2(\Omega)$ satisfying $\delta\phi = 0$ on Γ_1. Hence, instead of solving for ϕ satisfying the system of equations {(1), (2), (3)} we can minimize (4) subjected to the condition (2). This is the basis for the Finite Element Method.

Thus the Finite Element Method replaces the function $\phi \in C^2(\Omega)$ by an approximation function $\Phi(\underset{\sim}{x})$ which is determined by a finite number of its nodal values $\Phi(x_m)$, $m = 1, N$ inside Ω. These nodal values are determined by the minimization of $I(\Phi)$. The non-nodal values $\Phi(\underset{\sim}{x})$ are determined from the N nodal values so obtained. In all the following, the function $\Phi(\underset{\sim}{x})$ in each elemental volume will be a linear interpolation of its nodal values at the vertices of the element, as a result, the function $\Phi(x)$ satisfies $\nabla^2\Phi = 0$ within each elemental volume but not across inter-element boundaries (the higher order interpolation of $\Phi(x)$ does not automatically satisfy the condition $\nabla^2\Phi = 0$ within each element and additional equations are imposed as a result).

The functional $I(\Phi)$ is now discretized into

$$I(\Phi) = \int_{\Omega - \bigcup_{i=1}^{6} V_i} |\underset{\sim}{\nabla}\Phi|^2 + \sum_{i=1}^{6} \int_{V_i} |\underset{\sim}{\nabla}\Phi|^2$$

$$- 2 \int_{\Gamma - (\gamma_1 \cup \gamma_2)} \Phi p \quad - \quad 2 \sum_{i=1}^{2} \int_{\gamma_i} \Phi\, p \qquad ..(6)$$

as in Figure 1.

Since $\Phi(\underset{\sim}{x})$ within each elemental volume can not be affected by any nodal values outside the element, we have the equations determining $\Phi(x_m)$ as in the following:

For a node α_1 inside Ω, i.e. not on the boundary Γ_2,

$$0 = \sum_{i=1}^{7} \left(\begin{array}{l} \text{real constant depending only on the} \\ \text{geometry of } (\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7) \end{array} \right) \Phi_{\alpha_i} \quad . \qquad ..(7)$$

For a node α_3 on the boundary Γ_2,

$$p\alpha_3 = \sum_{i=1,2,3,4} \left(\begin{array}{l} \text{real constant depending only on} \\ \text{the geometry of } (\alpha_3,\alpha_2,\alpha_1,\alpha_4) \end{array} \right) \Phi_{\alpha_i} \quad . \qquad ..(8)$$

For typical nodes α_2, α_7 belonging to the boundary Γ_1,

$$\begin{aligned} u_{\alpha_2} &= \Phi_{\alpha_2} \ , \\ u_{\alpha_7} &= \Phi_{\alpha_7} \ . \end{aligned} \qquad ..(9)$$

All equations of the kind (7), (8), (9) form a large system of n equations in n unknowns. The function $\phi(\underset{\sim}{x})$ is considered known when its approximation $\Phi(\underset{\sim}{x})$ can be shown to be reasonably close to it.

The right hand sides of equations (7) and (8) result from the differentiation of a homogeneous second degree polynomial in Φ_{α_i}. Therefore, we have

$$\sum_{i=1}^{N} \Phi_{\alpha_i} \frac{\partial}{\partial \Phi_{\alpha_i}} \int_{\Omega} |\underset{\sim}{\nabla}\Phi|^2 = 2 \int_{\Omega} |\underset{\sim}{\nabla}\Phi|^2 \geq 0 \ .$$

Hence the matrix $\underset{\sim}{A}$ of coefficients in the equations (7), (8) and (9) satisfies

$$\Phi_{\alpha_i} \ A_{\alpha_i \alpha_j} \ \Phi_{\alpha_j} \geq 0 \ , \qquad ..(10)$$

and is positive definite (it is easy to show that $\underset{\sim}{A}$ is also symmetric). The Gauss-Seidel iteration method can thus be used to advantage in solving for the unknowns of the system {(7),(8),(9)}, see the appendix at the end of the memo.

The calculations in this section also hold if the function $\phi(\underset{\sim}{x})$ is complex-valued. In this case the last integral of equation (4) changes to Real $(\int_\Gamma \phi\, \bar{p})$ where $\bar{p}$ is the complex conjugate of p, the matrix $\underset{\sim}{A}$ of equation (10) becomes a positive definite Hermitian matrix, also equations (5) and (6) change slightly but this does not alter our final system {(7),(8),(9)}.

3. FORMULATION OF THE BOUNDARY ELEMENT METHOD

To solve for the solution of the same system {(1),(2),(3)} the Boundary Element Method starts with the identity

$$\int_\Omega (\nabla^2 \phi)\, y = 0 \qquad ..(11)$$

where y is any $C^2(\Omega)$ function. Using Green's theorem to integrate the above integral by parts, we have

$$\int_\Omega \phi\, (\nabla^2 y) - \int_\Gamma \phi\, \frac{\partial y}{\partial n} + \int_\Gamma y\, \frac{\partial \phi}{\partial n} = 0\,. \qquad ..(12)$$

Letting $\nabla^2 y = -\delta(\underset{\sim}{x} - \underset{\sim}{x}^*)$ where $\underset{\sim}{x}^*$ is on the boundary Γ of the region Ω, we have the following singular equation

$$\gamma\, \phi(\underset{\sim}{x}^*) + \int_{\Gamma - S(x^*)} \phi\, \frac{\partial y}{\partial n} - \int_{\Gamma - S(x^*)} y\, \frac{\partial \phi}{\partial n} = 0 \qquad ..(13)$$

where $S(\underset{\sim}{x}^*)$ is a very small sphere centered on $\underset{\sim}{x}^* \in \Gamma$ and γ is a constant equal to the fraction of the surface of $S(\underset{\sim}{x}^*)$ contained in Ω; when Γ is smooth this ratio γ is equal to one half.

The function $\phi \in C^2(\Omega)$ is now approximated by a function $\Psi(\underset{\sim}{x})$ which is determined by its nodal values $\Psi(\underset{\sim}{x}_m)$, $m = 1, M$ on the boundary Γ. The function Ψ is harmonic and its value on each elemental boundary element can be taken to be constant or the interpolation between the nodal values at the vertices of the (boundary) element. The equations determining the nodal values of Ψ are thus

$$\gamma\, \Psi(\underset{\sim}{x}_i) + \sum_{j=1}^{n} \int_{\Gamma_j - S(\underset{\sim}{x}_i)} \Psi\, \frac{\partial y_i}{\partial n} - \sum_{j=1}^{n} \int_{\Gamma_j - S(\underset{\sim}{x}_i)} y_i\, \frac{\partial \Psi}{\partial n} = 0$$

$$i = 1, \ldots, M \qquad ..(14)$$

where $y_i(\underset{\sim}{x})$ is given by

$$y_i(\underset{\sim}{x}) = \frac{1}{4\Pi} \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}_i|} \qquad i = 1,\ldots M. \qquad ..(15)$$

In any case the equations (14) have only M unknowns since either $\Psi(\underset{\sim}{x}_i)$ or $\frac{\partial \Psi}{\partial n}(\underset{\sim}{x}_i)$ is known for a given node.

For nodes $\underset{\sim}{x}_i$ on Γ_1, we have

$$G_{ij} \frac{\partial \Psi}{\partial n}(\underset{\sim}{x}_i) = H_{ij} \Psi(\underset{\sim}{x}_i) , \qquad i = 1,\ldots,M_1.$$

For nodes $\underset{\sim}{x}_i$ on Γ_2

$$H_{ij} \Psi(\underset{\sim}{x}_i) = G_{ij} \frac{\partial \Psi}{\partial n}(\underset{\sim}{x}_i) \qquad i = M_1 + 1,\ldots,M. \qquad ..(16)$$

The unknows on the right hand side of equations (16) are then solved and the value of $\Psi(\underset{\sim}{\xi})$ at any interior point $\underset{\sim}{a}$ of Ω is easily worked out using equation (12) with $\nabla^2 y = -\delta(\underset{\sim}{x} - \underset{\sim}{a})$.

Note that the approximation function $\Psi(\underset{\sim}{x})$ so derived satisfies equation (1) everywhere but satisfies the boundary conditions (2) and (3) only at M points. In other words, equation (11) is always satisfied inside Ω with $\Psi(\underset{\sim}{\xi})$ while equation (13) on the boundary Γ, which should also be satisfied identically for every singular function $y(\underset{\sim}{\xi}) = 1/|\underset{\sim}{x} - \underset{\sim}{x}^*|$, $x^* \in \Gamma$, is satisfied for only M functions $y_i(\underset{\sim}{x})$ defined by equation (15). For this reason the method has been considered by some authors a weighted residual method, meaning that equation (13) is satisfied only for a finite number (M, in this case) of weighting functions $y_i(\underset{\sim}{x})$.

It is now obvious that the Boundary Element Method depends on the use of Green's identity

$$\int_\Omega (a\nabla^2 b - b\nabla^2 a) \qquad \int_\Gamma \left(a \frac{\partial b}{\partial n} - b \frac{\partial a}{\partial n} \right) \qquad ..(17)$$

although the usual approach using the weighted residual method often disguises this under the process of partial integrations. Another interesting feature of the Boundary Element Method is the applicability of the Maximal Value Theorem which states that a harmonic function attains its maximal values only on the boundary [5]. The value of the methodical error $|\phi(x) - \Psi(x)|$ at an interior point of Ω is thus always less than such value at some point on the boundary Γ and the latter can be reduced with a finer discretization of Γ.

4. ERROR ANALYSIS FOR THE BOUNDARY AND FINITE ELEMENT METHODS

The determination of the maximum error $|\phi(x) - \Psi(x)|$ or $|\phi(x) - \Phi(x)|$ in Boundary and Finite Element Method is not at all simple. Although the Maximal Value Theorem gives an upper limit for the error function in the Boundary Element Method we are unlikely to know the limit unless $\Gamma_1 = \Gamma$.

4.1 Boundary Element Method

Let us first examine the simpler case of the Boundary Element Method. We can use Green's identity to write

$$4\Pi\ [\phi(x) - \Psi(x)] = -\int_{\Gamma} (\phi - \Psi)\left(\frac{\underset{\sim}{r}.\underset{\sim}{n}}{r^3}\right) + \int_{\Gamma} \frac{1}{r}\left(\frac{\partial\phi}{\partial n} - \frac{\partial\Psi}{\partial n}\right) \qquad ..(18)$$

we can say that the point error at each point depends mostly on the error on the boundary points closest to it and the variation is more prominent with $(\phi - \Psi)$ than with $\left(\frac{\partial\phi}{\partial n} - \frac{\partial\Psi}{\partial n}\right)$.

To remove all the unknown quantities such as $(\phi - \Psi)$ on Γ_2 and $\left(\frac{\partial\phi}{\partial n} - \frac{\partial\Psi}{\partial n}\right)$ on Γ_1 we have to use a Green's function $G(\underset{\sim}{y})$ satisfying

$$\nabla^2 G(\underset{\sim}{y}) = \delta(\underset{\sim}{y} - \underset{\sim}{x}) ,$$

$$G(\underset{\sim}{x}) = 0 \quad \text{on } \Gamma_1$$

and $$\frac{\partial G}{\partial n} = 0 \quad \text{on } \Gamma_2 . \qquad ..(19)$$

A new relation is then arrived at

$$4\Pi\ [\phi(x) - \Psi(x)] = -\int_{\Gamma_1} (\phi - \Psi)\frac{\partial G}{\partial n} + \int_{\Gamma_2} G\left(\frac{\partial\phi}{\partial n} - \frac{\partial\Psi}{\partial n}\right) . \qquad ..(20)$$

Obviously, a new function $G(\underset{\sim}{x})$ has to be calculated but the error $[\phi(x) - \Psi(x)]$ can be determined without any ambiguity. (The pole $\underset{\sim}{y}$ of the function G has been assumed to be in the interior of Ω. If Γ is smooth and $\underset{\sim}{y} \in \Gamma$ then at least one of the integrals on the right hand side is improper and the left hand side must correspondingly be divided by two). Another inequality will also be given in the next subsection.

4.2 Finite Element Method

The estimation of error for the Finite Element Method is more complicated as we have here in addition to the interpolation error on the boundary another kind of error arising from the discretization of the function $\Phi(x)$. This makes its gradient $\nabla\Phi$ vary in a stepwise manner within Ω and the jumps occur at inter-element boundaries. The Maximal Value Theorem is not applicable in this case as Φ is not harmonic over the region Ω.

A natural way to measure the closeness of $\Phi(x)$ to $\phi(x)$ is to use the difference

$$I(\Phi) - I(\phi) = \int_{\Omega} \left(|\nabla\Phi|^2 - |\nabla\phi|^2 \right) - 2 \int_{\Gamma} (\Phi - \phi)\, p \,. \qquad ..(21)$$

Hence

$$I(\Phi) - I(\phi) = 2 \int_{\Omega} \nabla\Phi \,.\, \nabla(\Phi - \phi) - \int_{\Omega} [\, \nabla(\Phi - \phi)\,]^2$$

$$- 2 \int_{\Gamma} (\Phi - \phi)\, p \,. \qquad ..(22)$$

Hence to have more accurate results we must make $\nabla(\Phi - \phi)$ small especially when $\nabla\Phi$ is large and similarly for $(\Phi - \phi)$ on the boundary, it must be small where p is large.

Similarly to the case of Boundary Element Method, the error in Finite Element Method can be written down as

$$4\pi\, [\phi(x) - \Phi(x)] = -\int_{\Gamma_1} (\phi - \Phi)\, \frac{\partial G}{\partial n} + \int_{\Gamma_2} G \left(\frac{\partial \phi}{\partial n} - \frac{\partial \Phi}{\partial n} \right) - \int_{S} G\, \frac{\partial \Phi}{\partial n} \qquad ..(23)$$

where G is the Green's function defined by equation (19) and S is a surface comprised of the double layers enclosing all inter-element boundaries with the normals pointing away from the elemental volumes. The equation holds for any point x interior to Ω. If x is right on the boundary Γ then at least one of the first two integrals on the right hand side of (23) is improper and also the left hand side must be divided by two. The last integral in equation (23) is the error caused by the stepwise variation of $\nabla\Phi$. Therefore, the errors $(\phi - \Phi)$ and $\left(\frac{\partial \phi}{\partial n} - \frac{\partial \Phi}{\partial n} \right)$ on Γ_1 and Γ_2 respectively, as well as the jump in $\nabla\Phi$ across inter-element boundaries must all be reduced if we want to reduce the magnitude of the left hand side of (23).

An often suggested measurement for the global performance (i.e. performance over the whole region Ω in an average manner) of a Finite Element scheme is the functional J defined by

$$J(\phi - \Phi) = \int_{\Gamma_1} | \phi - \Phi |^2 + \int_{\Gamma_2} \left| \frac{\partial \phi}{\partial n} - \frac{\partial \Phi}{\partial n} \right|^2 . \qquad ..(24)$$

In the following we will examine the usefulness of this functional.

From equations (23) and (24) we have an inequality

$$|\phi(x) - \Phi(x)| \leq \left| \int_S G \frac{\partial \phi}{\partial n} \right| + J^{\frac{1}{2}} \left\{ \int_{\Gamma_1} \left| \frac{\partial G}{\partial n} \right|^2 + \int_{\Gamma_2} |G|^2 \right\}^{\frac{1}{2}} \qquad ..(25)$$

which gives the upper bound for the point error $[\phi(x) - \Phi(x)]$ in terms of J and the Green's function defined by equation (19). It is obvious that even if we are only concerned with the resulting values of $\Phi(x)$ and $\phi(x)$ on the boundary Γ for a particular Finite Element scheme, the error originating from inter-element discontinuities in $\nabla\Phi$ still has to be accounted for. Hence the performance of the scheme can not be adequately measured by the use of J alone.

On the other hand, if we omit the first term on the right hand side of equation (25) and replace $\Phi(x)$ by $\Psi(x)$ we have an inequality for the Boundary Element Method. Hence J is a good measurement of the global performance of any Boundary Element scheme.

J can also be related to the quantity $[I(\Phi) - I(\phi)]$ by derivation from equations (21) and (24). We get

$$| I(\Phi) - I(\phi) | \leq \left| \int_S (\Phi - \phi) \frac{\partial \Phi}{\partial n} \right| +$$

$$+ J^{\frac{1}{2}} \left\{ \int_{\Gamma_1} \left| \frac{\partial \Phi}{\partial n} + \frac{\partial \phi}{\partial n} \right|^2 + \int_{\Gamma_2} | \Phi - \phi |^2 \right\}^{\frac{1}{2}} \qquad ..(26)$$

The first integral is the effect of the jump in $\nabla\Phi$ across inter-element boundaries and the last integral becomes negligible when Φ is close to ϕ.

In a similar way we also have

$$| I(\Phi) - I(\phi) | \leq \left\{ \int_S | \Phi - \phi |^2 + \int_{\Gamma} | \Phi - \phi |^2 \right\}^{\frac{1}{2}} *$$

$$* \left\{ \int_S \left| \frac{\partial \Phi}{\partial n} \right|^2 + \int_{\Gamma_1} \left| \frac{\partial \Phi}{\partial n} + \frac{\partial \phi}{\partial n} \right|^2 + \int_{\Gamma_2} \left| \frac{\partial \Phi}{\partial n} - p \right|^2 \right\}^{\frac{1}{2}} , \qquad ..(27)$$

which shows the effect of $|\Phi - \phi|$ on the difference $I(\Phi) - I(\phi)$.

5. RELATIVE COMPUTATIONAL ADVANTAGES OF FINITE AND BOUNDARY ELEMENT METHOD AS FOR THE TEST PROBLEM

The case where the region Ω is one dimensional is much too simple to have any significance. Therefore, only the cases where Ω is two or three-dimensional are considered.

First when Ω is two-dimensional the FEM (Finite Element Method) has n^2 nodes and requires n^2 memory locations. In contrast, the BEM (Boundary Element Method) requires only n nodes but also n^2 memory locations. If the Gauss-Seidel iterative method is used for the banded matrix of the FEM and the standard Doolittle or Gaussian method is used for the full natrix of the BEM then their computing time will be proportional to n^4 and n^3 respectively. Thus the BEM is advantageous in this case.

On the other hand, when Ω is three-dimensional, the FEM and BEM requre(s) n^3 and n^2 nodes respectively. Memory locations required are of order n^3 and n^4 respectively and the computing times are the same. Thus, the FEM is advantageous in this case.

6. FINAL REMARKS

Besides those points considered in the previous section, others which favours one method over the other are:

(i) The gradient of $\phi(x)$ is approximated by step-like function $\nabla\Phi$ in FEM compared to a harmonic function $\nabla\Psi$ in BEM.

(ii) The setting up of a BEM program requires some elaborate calculation for each singular component of the surface integral. This is in direct contrast with the straightforward setting up procedure of FEM. Moreover, the values of the integrals in BEM are fairly sensitive to the accuracy of the integration near to their singularities. This creates yet another kind of error which has to be considered.

(iii) For problems with a non-homogenous body the BEM requires a subdivision of the body while FEM has this feature inherent in it.

(iv) With any subdivision of a body such as in (iii) or as for a long a narrow region (for example, see [1], p. 183) the BEM becomes a hybrid between itself and the FEM. Mixed characters can thus be expected in this case : For example, the coefficient matrix becomes slightly banded and the function $\Psi(x)$ is not harmonic at all interior point of the region Ω.

(v) The truncation error in the Gauss-Seidel iteration for the FEM is usually of greater order of magnitude than the round off error. The latter also does not improve much with the use of internal double precision in its equation solving subroutine.

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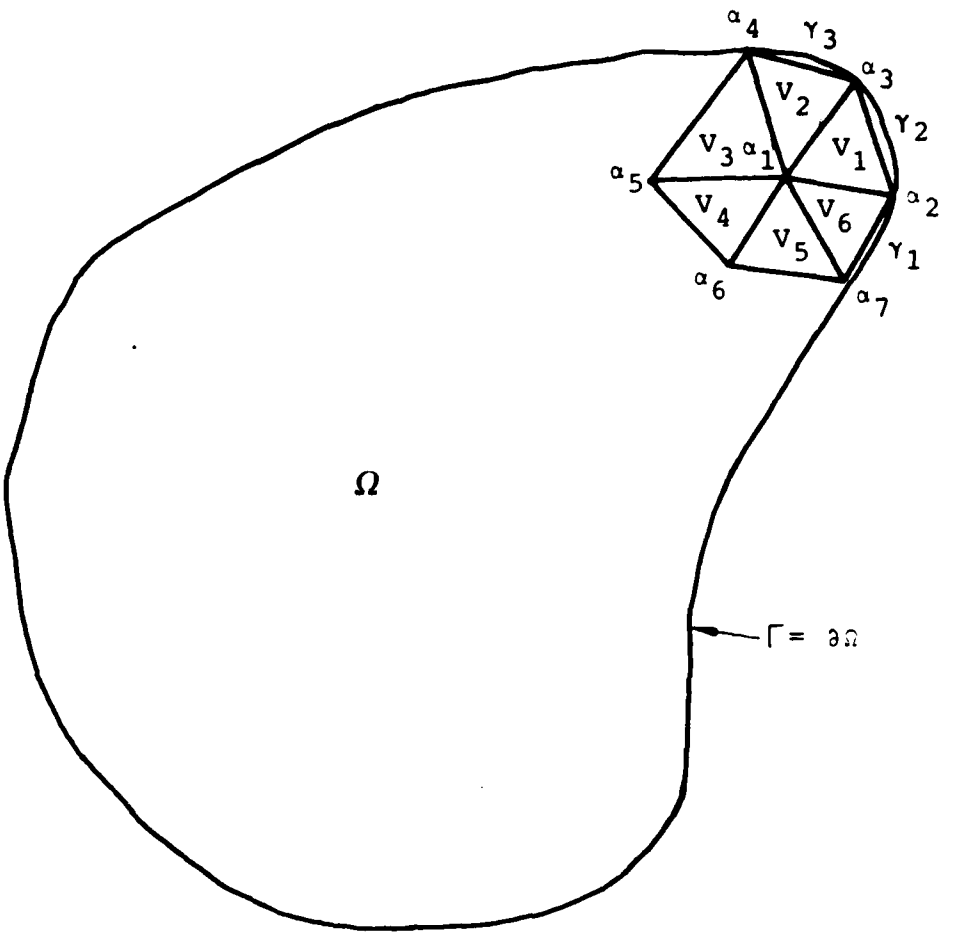

$\alpha_2, \alpha_7 \in \gamma_1 \subset \Gamma$

$\alpha_2, \alpha_3, \alpha_4 \in \gamma_2 \cup \gamma_3 \subset \Gamma_2$

FIG. 1 DISCRETIZATION OF A REGION Ω AND ITS BOUNDARY Γ

APPENDIX

Convergence of Gauss-Seidel Method for Positive Definite Hermitian Matrices

To make the memo self-contained, a proof of convergence for the Gauss-Seidel method as applied to a positive definite Hermitian matrix is given here. This proof is a generalization of the one given in [7], p. 445 for a real symmetric matrix. The convergence is independent of the initial vector.

Consider the system of equation

$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$$

where $\underset{\sim}{A}$ is a positive definite Hermitian matrix.

Decompose $\underset{\sim}{A}$ into

$$\underset{\sim}{A} = \underset{\sim}{L} + \underset{\sim}{D} + \underset{\sim}{U}$$

where $\underset{\sim}{L}$ and $\underset{\sim}{U}$ are strictly lower and upper complex matrice and $\underset{\sim}{D}$ a diagonal positive definite real matrix. The Gauss-Seidel iteration method then gives

$$\underset{\sim}{x}_{i+1} = -(\underset{\sim}{D}+\underset{\sim}{L})^{-1}\,\underset{\sim}{U}\,\underset{\sim}{x}_i + (\underset{\sim}{D}+\underset{\sim}{L})^{-1}\,\underset{\sim}{b}\ .$$

Hence,

$$\underset{\sim}{x}_{i+1} - \underset{\sim}{x}_i = -(\underset{\sim}{D}+\underset{\sim}{L})^{-1}\,\underset{\sim}{U}\,(\underset{\sim}{x}_i - \underset{\sim}{x}_{i-1})$$

The sequence $\{\underset{\sim}{x}_i\}$ converges if all the eigenvalues of the matrix $\underset{\sim}{B} = -(\underset{\sim}{D}+\underset{\sim}{L})^{-1}\,\underset{\sim}{U}$ have their magnitudes smaller than unity.

Since $\underset{\sim}{A}$ is Hermitian we have

$$\underset{\sim}{B} = -(\underset{\sim}{D}+\underset{\sim}{L})^{-1}\,\overline{\underset{\sim}{L}}^{T}$$

where the overbar denotes the complex conjugate. Let λ and $\underset{\sim}{v}$ be an eigenvalue and eigenvector of the matrix $\underset{\sim}{B}$. We will prove that $|\lambda| < 1$.

Since,

$$\bar{L}^T \underset{\sim}{v} = \lambda(\underset{\sim}{D} + \underset{\sim}{L})\underset{\sim}{v} \quad ,$$

or

$$(1 + \lambda)\, \bar{\underset{\sim}{v}}\, (\underset{\sim}{D} + \underset{\sim}{L})\underset{\sim}{v} = \bar{\underset{\sim}{v}}\, \underset{\sim}{A}\, \underset{\sim}{v} > 0 \quad ,$$

λ is different from -1.

Taking the conjugate transpose of the above equation we have

$$(1 + \lambda)\, \bar{\underset{\sim}{v}}\, (\underset{\sim}{D} + \underset{\sim}{L})\, \underset{\sim}{v} = (1 + \bar{\lambda})\, \bar{\underset{\sim}{v}}\, (\underset{\sim}{D} + \overline{\underset{\sim}{L}^T})\, \underset{\sim}{v}$$

$$= (1 + \bar{\lambda})\, \bar{\underset{\sim}{v}}\, \underset{\sim}{D}\, \underset{\sim}{v} + \lambda(1 + \bar{\lambda})\, \bar{\underset{\sim}{v}}\, (\underset{\sim}{D} + \underset{\sim}{L})\, \underset{\sim}{v},$$

or

$$(1 - |\lambda|^2)\, \bar{\underset{\sim}{v}}\, (\underset{\sim}{D} + \underset{\sim}{L})\, \underset{\sim}{v} = (1 + \bar{\lambda})\, \bar{\underset{\sim}{v}}\, \underset{\sim}{D}\, \underset{\sim}{v}$$

So,

$$(1 - |\lambda|^2)\, \bar{\underset{\sim}{v}}\, \underset{\sim}{A}\, \underset{\sim}{v} = (1 + \lambda)\, (1 + \bar{\lambda})\, \bar{\underset{\sim}{v}}\, \underset{\sim}{D}\, \underset{\sim}{v} \, .$$

The right hand side is greater than zero since $\underset{\sim}{D}$ is positive definite. Thus $|\lambda| < 1$, which proves the convergence of the iteration scheme.

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The Finite Element and Boundary Element Methods are described with their essential features illustrated using an example of a boundary value problem for a harmonic function. Analysis of the methodical errors is then carried out. This is followed by a consideration of the relative computational advantages of the two methods.

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